Indian Statistical Institute, Bangalore Centre
B.Math. (III Year) : 2011-2012
Semester II : Semestral Examination
Probability III (Stochastic Processes)

9.5.2012 Time: 3 hours. Maximum Marks : 100

Note: The paper carries 105 marks. Any score above 100 will be taken as 100. State clearly the results you are using in your answers.

1. (7+8+5 = 20 marks) A particle moves according to a Markov chain on $\{1, 2, \cdots, c + d\}$ where c, d are positive integers. Starting from any one of the first c states, the particle jumps in one transition to a state chosen uniformly from the last d states; starting from any one of the last d states, the particle jumps in one transition to a state chosen uniformly from the first c states.

   (i) Show that the Markov chain is irreducible and recurrent.

   (ii) Find the stationary probability distribution. Is the Markov chain positive recurrent?

   (iii) Does the Markov chain have a limiting distribution?

2. (15 marks) Let $M > 0$ be an integer. Let $\{X_n : n = 0, 1, 2, \cdots\}$ be a Markov chain on $\{0, 1, \cdots, M\}$ such that (i) $0, M$ are absorbing states, and (ii) $P_{i,i+1} = p_i > 0, P_{i,0} = (1 - p_i) > 0,\ 1 \leq i \leq (M - 1)$. For $1 \leq k \leq (M - 1)$ find the probability of absorption at 0 starting from k.

3. (25 marks) $\{X_n : n = 0, 1, 2, \cdots\}, \{Y_n : n = 0, 1, 2, \cdots\}$ are independent, irreducible, aperiodic, positive recurrent Markov chains on a countable state space S with the same transition probability matrix $P = ((P_{ij}))$. Let $T = \min\{n \geq 1 : X_n = Y_n\}$. Show that $T < \infty$ with probability one. (Hint: Consider $\{(X_n, Y_n) : n \geq 0\}$.)

4. (15 marks) Suppose that customers arrive at a bus stop in accordance with a time homogeneous Poisson process with arrival rate $\lambda > 0$. Suppose the bus departs at a time $T > 0$ which is fixed. Find the expected sum of waiting times of customers who manage to board the bus.

5. (15 marks) Let $\{N(t) : t \geq 0\}$ be a non-homogeneous Poisson process with a continuous intensity function $\lambda(\cdot)$. Let $0 < t_1 < t_2 < \infty$. Find $P(N(t_1) = k \mid N(t_2) = n)$ for nonnegative integers k, n with $k \leq n$.

6. (6+9 = 15 marks) Let $\{X(t) : t \geq 0\}$ be a pure birth process on state space $S = \{0, 1, 2, \cdots\}$ with infinitesimal birth rates $\lambda_i, i \in S$.

   (i) Using forward Kolmogorov equations, find $P(X(t) = i \mid X(0) = i), i \in S$.

   (ii) For any $k \in S$ find the distribution of the holding time $\tau_k = \inf\{t > 0 : X(t) \neq k\}$, assuming $X(0) = k$.